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On a sufficient condition for the existence of N-particle bound states

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Abstract. We use a Birman-Schwinger type analysis to derive a sufficient condition for the existence of bound states in a certain class of N-body systems. This condition is given in terms of the underlying subsystem with the threshold energy $\Sigma = \inf \sigma_{ess}(H)$.

1. Introduction

In 1961 Birman and Schwinger introduced a method to control the number of bound states in a two-body system in three dimensions. Based on the fact that the eigenvalues of the Schrödinger Hamiltonian h for reasonable potentials v with positive coupling constant λ are continuous and monotonically decreasing functions of λ, they noted that the number of eigenvalues of h (counting multiplicity) below a fixed negative energy $-\kappa^2$, equals the number of eigenvalues $\lambda^{-1} > 1$ of the so-called Birman-Schwinger kernel $|v|^{1/2}(-\Delta+\kappa^2)^{-1}|v|^{1/2}$. (We use natural units with $\hbar^2/2m = 1$.) This kernel is Hilbert-Schmidt, even in the uniform limit $-\kappa^2 \uparrow 0$. Hence the number of discrete eigenvalues of the two-body Hamiltonian h is determined by the number of eigenvalues $\lambda^{-1} > 1$ of $|v|^{1/2}(-\Delta)^{-1}|v|^{1/2}$ where the multiplicities of the corresponding eigenvalues of h and the Birman-Schwinger kernel are exactly the same. There exist rather explicit bounds on that number (see, e.g., Reed and Simon (1978) or Simon (1971) for details).

Klaus and Simon (1980a) extended these arguments to the case of a N-body system of ν-dimensional particles with unique two-cluster threshold $\Sigma = \inf \sigma_{ess}(H) < 0$, where H denotes the N-body Hamiltonian and $\sigma_{ess}(H)$ is the essential spectrum of H. Sigal (1983a, b) also generalised the Birman-Schwinger principle to the case of specific N-particle systems.

In this paper we use the above mentioned results for three dimensions to derive a sufficient condition for the existence of at least one N-body bound state below the threshold Σ. This condition is a direct generalisation of the corresponding two-body result given by Chadan and Martin (1980) (see also Chadan and De Mol 1980). We

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also mention that sufficient conditions for the existence of bound states of N particles with attractive potentials have been derived using the variational principle (Coutinho *et al* 1984).

In § 2 we briefly outline the technical devices to be used. Section 3 derives the sufficient condition for the existence of at least one N-body bound state below a unique two-cluster threshold, determined by the ground-state energy of a $(N-1)$-particle subsystem, for $N\geqslant 3$ particles. The restriction to break-up into a $(N-1)$-particle cluster and a 'free' particle is just for convenience. Any two-cluster break-up can be treated in the same way. Finally in § 4 we briefly discuss the corresponding problem for N-body systems ($N\geqslant 3$) with none of the subsystems having a bound state or resonance at zero energy.

2. Birman–Schwinger kernels

We consider a N-body system of three-dimensional distinguishable particles described by the Hamiltonian H

$$H = H_0 + V \tag{2.1}$$

acting in $\mathscr{H} = L^2(\mathbb{R}^{3(N-1)})$, where H_0 denotes the free Hamiltonian after the centre-of-mass motion has been eliminated and where V is the sum of pairwise potentials v_{ij}

$$V = \sum_{i<j} v_{ij}(r_i - r_j). \tag{2.2}$$

Throughout the paper the potentials are assumed to satisfy

$$v_{ij} \leqslant 0 \tag{2.3}$$

and

$$v_{ij} \in C_0^\infty(\mathbb{R}^3). \tag{2.4}$$

We follow the notation of Reed and Simon (1979). In particular we use clustered Jacobi coordinates $\{\xi_1, \ldots, \xi_{N-2}, \zeta_D\}$. We assume that there exists a unique partition D of the N particles into a $(N-1)$ cluster and a 'free' particle defining in this way a two-cluster break-up of the system. Let Σ_{iDj} (resp $\Sigma_{\sim iDj}$) denote the sum over all particles with (i, j) in the same (resp different) cluster of D. Then we define

$$V_D = \sum_{iDj} v_{ij} \qquad I_D = \sum_{\sim iDj} v_{ij} \tag{2.5}$$

$$H_D = H_0 + V_D = H - I_D. \tag{2.6}$$

For this partition D one may decompose $\mathscr{H}$ as $\mathscr{H}_D \otimes \mathscr{H}^D \equiv L^2(\mathbb{R}^{3(N-2)}, \mathrm{d}\xi_D) \otimes L^2(\mathbb{R}^3, \mathrm{d}\zeta_D)$ where $\xi_D = (\xi_1, \xi_2, \ldots, \xi_{N-2})$. Then

$$H_D = h_D \otimes 1 + 1 \otimes (-\Delta_{\zeta_D}) \tag{2.7}$$

with h_D denoting the $(N-1)$-particle Hamiltonian. The latter determines the equation

$$h_D \psi_D = \Sigma_D \psi_D \tag{2.8}$$

where ψ_D is the $(N-1)$-particle ground state. We need the notion of a unique two-cluster threshold: we call the partition D unique two-cluster if the threshold $\Sigma = \inf \sigma_{\mathrm{ess}}(H) = \Sigma_D < \Sigma_{D'}$ for all possible partitions D and D' of the N-particle system.

Following Klaus and Simon (1980a) we define the two-cluster Birman–Schwinger kernel $K_D(E)$ by

$$K_D(E) = |I_D|^{1/2}(H_D - E)^{-1}|I_D|^{1/2} \tag{2.9}$$

for $E < \Sigma_D = \inf \sigma_{\text{ess}}(H)$. We then recall the following results of Klaus and Simon (1980a).

Lemma 2.1. Let H describe a N-body system with potentials satisfying (2.3) and (2.4). Let $\Sigma_D < 0$ be a unique two-cluster threshold of H. Then $K_D(E)$ has a norm-limit $K_D(\Sigma_D)$ as $E \uparrow \Sigma_D$.

Lemma 2.2. Let H be the Hamiltonian of a N-particle system with potentials obeying (2.3) and (2.4) and let $\Sigma_D < 0$ be a unique two-cluster threshold. Then the number $N(H)$ of discrete eigenvalues of H (counting multiplicity) is finite and

$$N(H) = \#\{\lambda > 1,\ \lambda \text{ an eigenvalue of } K_D(\Sigma_D)\}.$$

In contrast to the two-body case $K_D(E)$ is no longer compact. But for a unique two-cluster threshold the Birman–Schwinger principle as expressed in lemma 2.2 remains valid.

We now want to use this principle to arrive at a sufficient condition for the existence of at least one N-body bound state. We start by looking at the following Schrödinger equation (valid at least in the sense of distributions) for a unique two-cluster break-up:

$$(H_0 + V_D + I_D)\Psi = \Sigma_D \Psi. \tag{2.10}$$

We know that either Ψ is a bound state (i.e. $\Psi \in L^2(\mathbb{R}^{3(N-1)})$ and $\Sigma_D \in \sigma_{\text{pp}}(H)$ where $\sigma_{\text{pp}}(H)$ is the pure point spectrum of H) or Ψ is a resonance at threshold energy (i.e. $\Psi \notin L^2(\mathbb{R}^{3(N-1)})$ (Klaus and Simon 1980a)) or Ψ is a scattering solution (i.e. Ψ is oscillating for $|\zeta_D| \to \infty$ and $E > \Sigma_D$). From (2.10) we then obtain

$$|I_D|^{1/2}\Psi(\Sigma_D) = |I_D|^{1/2}\psi_D + |I_D|^{1/2}(H_D - \Sigma_D)^{-1}|I_D|^{1/2}|I_D|^{1/2}\Psi(\Sigma_D). \tag{2.11}$$

The inhomogeneous term $|I_D|^{1/2}\psi_D \neq 0$ stems from the unique solution of

$$(h_D - \Delta_{\zeta_D})\psi_D(\xi_D)\exp[\mathrm{i}(E - \Sigma_D)^{1/2}\zeta_D] = E\psi_D(\xi_D)\exp[\mathrm{i}(E - \Sigma_D)^{1/2}\zeta_D] \tag{2.12}$$

at threshold Σ_D. (We omit the Σ_D dependence of Ψ in the following.)

Therefore $|I_D|^{1/2}\Psi$ defined in (2.11) can be expressed by an absolutely convergent series (Born series) as long as $\|K_D(\Sigma_D)\| < 1$ or, in other words, as long as there are no N-body bound states below threshold. So to obtain the sufficient condition for having at least one N-body bound state we have to determine when exactly this iteration procedure breaks down.

3. Two-cluster break-up: a sufficient condition for having at least one bound state

We now study (2.11) in detail:

$$\begin{aligned}
&|I_D(X_D)|^{1/2}\Psi(X_D)\\
&\qquad = |I_D(X_D)|^{1/2}\psi_D(\xi_D)\\
&\qquad\quad + |I_D(X_D)|^{1/2}\int \mathrm{d}X'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\Psi(X'_D)
\end{aligned} \tag{3.1}$$

where $\Sigma_D < 0$ is a unique two-cluster threshold, X_D abbreviates (ξ_D, ζ_D) and $G_D(X_D, X'_D; \Sigma_D)$ denotes the Green function corresponding to H_D. We prove the following.

Lemma 3.1. Suppose $\|K_D(\Sigma_D)\| < 1$ and $\Sigma_D < 0$ is a unique two-cluster threshold of a $(N \geq 3)$-body system. Then $[1 - K_D(\Sigma_D)]^{-1}$ is positivity preserving and $|I_D|^{1/2}\Psi$, as well as Ψ defined in (2.11), is positive.

Proof. We note that $H_0 \equiv -\Delta_{X_D}$. Hence $\exp(-tH_0) \equiv \exp(t\Delta_{X_D})$ is positivity improving for all $t > 0$ (see Reed and Simon 1978, ch XIII.12, example 1). Therefore we deduce from theorem XIII.45 and the proposition on p 204 of Reed and Simon (1978) that $(H_D - E)^{-1}$ is positivity preserving for all $E < \Sigma_D$ (V_D is bounded, see (2.4)). Thus $K_D(E)\phi$ is positive or the zero function for all positive $\phi \in L^2(\mathbb{R}^{3(N-1)})$ and for $E < \Sigma_D$. The strong convergence of $K_D(E) \to K_D(\Sigma_D)$ as $E \uparrow \Sigma_D$ (see lemma 2.1) implies that also $K_D(\Sigma_D)\phi$ is positive or the zero function for all positive ϕ: $K_D(\Sigma_D - 1/n)\phi \to K_D(\Sigma_D)\phi$ in $L^2(\mathbb{R}^{3(N-1)})$ as $n \to \infty$ determines the existence of a subsequence $\{[K_D(\Sigma_D - 1/n_j)\phi]\}_{j \in \mathbb{N}}$ such that $[K_D(\Sigma_D - 1/n_j)\phi](X_D) \to [K_D(\Sigma_D)\phi](X_D)$ almost everywhere as $j \to \infty$.

By assumption we have

$$[1 - K_D(\Sigma_D)]^{-1} = \sum_{n=0}^{\infty} [K_D(\Sigma_D)]^n \tag{3.2}$$

such that $[1 - K_D(\Sigma_D)]^{-1}\phi$ is positive for all positive $\phi \in L^2(\mathbb{R}^{3(N-1)})$. Furthermore from theorem XIII.46 of Reed and Simon (1978) we infer that $|I_D|^{1/2}\psi_D$ is positive and uniquely determined. Therefore it follows that

$$|I_D|^{1/2}\Psi = [1 - K_D(\Sigma_D)]^{-1}|I_D|^{1/2}\psi_D \tag{3.3}$$

and consequently Ψ is positive.

Next we consider the sphere S^{3N-4} with radius $|X_D| = R$, which divides the total space in two regions. We define

$$M_1 = \inf_{|X_D| \leq R} \Psi(X_D) \tag{3.4}$$

and

$$M_2 = \inf_{|X_D| > R} [|X_D|^{\nu}\Psi(X_D)] \tag{3.5}$$

with some positive constant ν to be specified in lemma 3.2 below. We denote by $\chi_M(X_D)$ the characteristic function of the set M. Then we have

$$\begin{aligned}\phi(X_D) = |I_D(X_D)|^{1/2}[\Psi(X_D) - \inf_{|X_D| \leq R} \Psi(X_D)\chi_{|X_D| \leq R}(X_D) \\ - |X_D|^{-\nu} \inf_{|X_D| > R} (\Psi(X_D)|X_D|^{\nu})\chi_{|X_D| > R}(X_D)] \geq 0\end{aligned} \tag{3.6}$$

almost everywhere. Therefore $K_D(\Sigma_D)\phi$ is positive or the zero function (see the proof of lemma 3.1). Furthermore, we recall that $|I_D|^{1/2}\psi_D$ is positive and ψ_D is strictly positive. Using this we obtain from (3.1)

$$\Psi(X_D) > \int_{|X'_D| \le R} dX'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)| \inf_{|X'_D| \le R} \Psi(X'_D)$$
$$+ \int_{|X'_D| > R} dX'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\, |X'_D|^{-\nu}$$
$$\times \inf_{|X'_D| > R} (\Psi(X'_D)|X'_D|^{\nu}). \tag{3.7}$$

Taking the infima on both sides, we arrive at

$$M_1 > M_1 \inf_{|X_D| \le R} \left(\int_{|X'_D| \le R} dX'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)| \right)$$
$$+ M_2 \inf_{|X_D| \le R} \left(\int_{|X'_D| > R} dX'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\, |X'_D|^{-\nu} \right) \tag{3.8}$$

as well as

$$M_2 > M_1 \inf_{|X_D| > R} \left(|X_D|^{\nu} \int_{|X'_D| \le R} dX'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)| \right)$$
$$+ M_2 \inf_{|X_D| > R} \left(|X_D|^{\nu} \int_{|X'_D| > R} dX'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\, |X'_D|^{-\nu} \right). \tag{3.9}$$

To proceed further we need the following lemma.

Lemma 3.2. Define for $\nu = 3N-5$ the functions F, f and g by

$$F(X_D) = M_1 \int_{|X'_D| \le R} dX'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|$$
$$+ M_2 \int_{|X'_D| > R} dX'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\, |X'_D|^{-\nu}$$
$$f(R) = \inf_{|X_D| = R} F(X_D) \qquad \text{and} \qquad g(R) = \inf_{|X_D| = R} [|X_D|^{\nu} F(X_D)].$$

Then f is decreasing in R and g is increasing in R.

Proof. We define $W \in L^2(\mathbb{R}^{3(N-1)})$ by

$$W(X_D) = |I_D(X_D)|[M_1 \chi_{|X_D| \le R}(X_D) + M_2 \chi_{|X_D| > R}(X_D)|X_D|^{-\nu}] \qquad \text{for } \nu = 3N-5 \tag{3.10}$$

and approximate W by a sequence of positive functions $W_n \in C_0^\infty(\mathbb{R}^{3(N-1)})$ such that $W = \lim_{n\to\infty} W_n$ in $L^2(\mathbb{R}^{3(N-1)})$. We note that $W_n \in \mathrm{Ran}(H_D - \Sigma_D)$ for all n since $(H_D - \Sigma_D)$ maps the set of C_0^∞ functions onto itself ($V_D = \Sigma_{iDj} v_{ij}$ and $v_{ij} \in C_0^\infty(\mathbb{R}^3)$). Furthermore, $\mathscr{D}(H_0) = \mathscr{D}(H_D - \Sigma_D)$. ($\mathscr{D}(T)$ and $\mathrm{Ran}(T)$ denote domain and range of the operator T.) Therefore we have

$$\Delta_{X_D} F_n = -H_0(H_D - \Sigma_D)^{-1} W_n \in L^2(\mathbb{R}^{3(N-1)})$$

for all n. Thus

$$\Delta_{X_D} F_n = -W_n - \Sigma_D(H_0 - \Sigma_D)^{-1} W_n - |V_D|(H_D - \Sigma_D)^{-1} W_n$$
$$- \Sigma_D(H_0 - \Sigma_D)^{-1}|V_D|(H_D - \Sigma_D)^{-1} W_n. \tag{3.11}$$

From $[-\Sigma_D(H_0-\Sigma_D)^{-1}]\leq 1$ in the sense of operators and (3.11) we obtain that $\Delta_{X_D}F_n\leq 0$, i.e. F_n is a superharmonic function. (A superharmonic function is minus a subharmonic function, see Hayman and Kennedy (1976, ch 2).) Then f_n defined as the infimum of a superharmonic function is also superharmonic (Hayman and Kennedy 1976, p 41). This implies f_n is decreasing in R (Hayman and Kennedy 1976, theorems 1.12 and 2.8).

The superharmonicity of f_n gives (F_n and therefore f_n are in the domain of H_0 for all n)

$$(\Delta_{X_D}f_n)(R)=R^{-(3N-4)}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^{(3N-4)}\frac{\mathrm{d}}{\mathrm{d}R}f_n(R)\right)\leq 0. \tag{3.12}$$

After multiplication by R and integration from $\hat{R}$ to infinity, inequality (3.12) becomes

$$(3N-5)f_n(\hat{R})+\hat{R}f'_n(\hat{R})\geq \lim_{R\to\infty}[(3N-5)f_n(R)+Rf'_n(R)]. \tag{3.13}$$

Because of the fact that f_n is positive, decreasing and $f'_n\in H^{2,1}(\mathbb{R}^{3(N-1)})$ we obtain that the RHS of inequality (3.13) vanishes for $R\to\infty$ ($H^{2,1}$ is the Sobolev space of order one).

As a consequence we deduce

$$\frac{\mathrm{d}}{\mathrm{d}R}R^{(3N-5)}f_n(R)\geq 0 \tag{3.14}$$

proving that $R^{(3N-5)}f_n(R)$ is increasing in R for all n.

The boundedness of $K_D(\Sigma_D)$ implies that $\int \mathrm{d}X'_D\, G_D(X_D, X'_D; \Sigma_D)\, W(X'_D)$ exists almost everywhere. Therefore it follows by dominated convergence that $F_n(X_D)\to F(X_D)$ as $n\to\infty$ almost everywhere. Hence F and f are decreasing functions and $R^{(3N-5)}f(R)$ is an increasing function of R.

As a consequence of lemma 3.2 the infima in inequalities (3.8) and (3.9) are reached at $|X_D|=R$. We set

$$\inf_{|X_D|=R}\left(\int_{|X'_D|\leq R}\mathrm{d}X'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\right)=J_1 \tag{3.15}$$

$$\inf_{|X_D|=R}\left(\int_{|X'_D|>R}\mathrm{d}X'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\,|X'_D|^{-\nu}\right)=J_2. \tag{3.16}$$

Then inequalities (3.8) and (3.9) are simply rewritten as

$$M_1>M_1J_1+M_2J_2 \tag{3.17}$$

$$M_2>R^{\nu}(M_1J_1+M_2J_2). \tag{3.18}$$

This implies

$$M_1J_1+M_2J_2>(J_1+R^{\nu}J_2)(J_1M_1+J_2M_2). \tag{3.19}$$

Therefore a necessary condition for the absence of discrete eigenvalues of H (i.e. convergence of the Born series for Ψ) is expressed by $J_1+R^{\nu}J_2<1$.

Throughout the exposition we discussed the break-up of the N-body system into a $(N-1)$-body subsystem and a 'free' particle. This restriction was assumed for convenience only. Since $\Sigma=\inf\sigma_{\mathrm{ess}}(H)<0$ is determined by a splitting of the system into two clusters (Reed and Simon 1978, p 122), our final result, as stated in theorem 3.3 below, extends to the most general situation.

Theorem 3.3. Suppose $\Sigma = \inf \sigma_{\text{ess}}(H) = \Sigma_D < 0$ to be a unique two-cluster threshold of the N-particle Hamiltonian H ($N \geqslant 3$) with potentials obeying (2.3) and (2.4). Then a sufficient condition for H to have a discrete eigenvalue is given by

$$\inf_{|X_D|=R}\left(\int_{|X'_D|\leqslant R} \mathrm{d}X'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\right) + R^\nu \inf_{|X_D|=R}\left(\int_{|X'_D|>R} \mathrm{d}X'_D\, G_D(X_D, X'_D; \Sigma_D)|I_D(X'_D)|\,|X'_D|^{-\nu}\right) > 1$$

for arbitrary R and $\nu = 3N-5$.

Replacement of the inequality in theorem 3.3 by an equality means that an eigenvalue or resonance of H just appears at threshold out of the continuous spectrum.

4. N-cluster threshold for ($N \geqslant 3$)-body systems

The threshold $\Sigma = 0$ is called N-cluster if none of the subsystems has a bound state or resonance at zero energy. There is no equivalent to (2.11) in $L^2(\mathbb{R}^{3(N-1)})$ for the N-cluster threshold. Nevertheless the homogeneous equation

$$|V|^{1/2}H_0^{-1}|V|^{1/2}(|V|^{1/2}\Psi) = \lambda^{-1}|V|^{1/2}\Psi \tag{4.1}$$

makes sense and the number of eigenvalues $\lambda^{-1} > 1$ of this N-cluster Birman–Schwinger kernel determines the number of discrete eigenvalues of H (Karner 1985).

Therefore the following conjecture seems to be justified. A sufficient condition for the existence of a bound state in a ($N \geqslant 3$)-body system with N-cluster threshold and potentials obeying (2.3) and (2.4) is

$$\inf_{|X|=R}\int_{|X'|\leqslant R} \mathrm{d}X'\, G_0(X, X'; 0)|V(X')| + R^\nu \inf_{|X|=R}\int_{|X'|>R} \mathrm{d}X'\, G_0(X, X'; 0)|V(X')|\,|X'|^{-\nu} > 1 \tag{4.2}$$

for arbitrary R and $\nu = 3N-5$ ($X = (\xi_1, \ldots, \xi_{N-1})$ are the Jacobi coordinates). The Green function $G_0(X, X'; 0)$ corresponding to H_0 is explicitly known (e.g. Jensen 1980).

(i) For N = even number of particles ($\mu = 3(N-1)$):

$$G_0(X, X'; 0) = 2^{(\mu-5)/2}(2\pi)^{-(\mu-1)/2}(\mu-3)!\{[(\mu-3)/2]!\}^{-1}|X-X'|^{-(\mu-2)}. \tag{4.3}$$

(ii) For N = odd number of particles ($\mu = 3(N-1)$):

$$G_0(X, X'; 0) = 4^{-1}\pi^{-\mu/2}(\mu/2-2)!\,|X-X'|^{-(\mu-2)}. \tag{4.4}$$

Finally we remark that $|V|^{1/2}H_0^{-1}|V|^{1/2}$ is bounded for $v_{ij} \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ with $q < \frac{3}{2} < p$ (e.g. Sigal 1983b). Hence the condition (4.2) is expected to hold for a large class of potentials.

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